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HYPERGEOMETRIC SOLUTIONS TO SCHRÖDINGER EQUATIONS FOR THE QUANTUM PAINLEVÉ EQUATIONS

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ABSTRACT. We consider Schrödinger equations for the quantum Painlevé equations. We present hypergeometric solutions of the Schrödinger equations for the quantum Painlevé equations, as particular solutions. We also give a representation theoretic correspondence between Hamiltonians of the Schrödinger equations for the quantum Painlevé equations and those of the KZ equation or the confluent KZ equations.

1. INTRODUCTION

Quantizations of the Painlevé equations with affine Weyl group symmetries, in the Heisenberg picture, were proposed and studied in [7], [11], [12], and [14]. We have called these quantizations, quantum Painlevé equations. Although it is well-known that the Painlevé equations admit exact solutions, such as hypergeometric solutions, algebraic or rational solutions, no exact solutions of the quantum Painlevé equations were given in the previous studies.

Aiming to construct exact solutions to the quantum Painlevé equations, together with M. Jimbo and J. Sun, we introduced confluent KZ equations for \mathfrak{sl}_2 , irregular singular versions of the KZ equations [7]. The confluent KZ equations for \mathfrak{sl}_2 have integral formulas for solutions, which take values in confluent Verma modules. We derived the quantum Painlevé equations in a formal algebraic way from the Heisenberg version of the confluent KZ equations. In the derivation process, we need to invert some elements. For the confluent KZ equations associated with highest weight-type modules, this invertibility fails. Hence the integral solutions to the confluent KZ equations do not give rise to solutions to the quantum Painlevé equations.

However, the work [7] gives us some insight into a possible link between Hamiltonians of the (confluent) KZ equations and those of Schrödinger equations for the quantum Painlevé equations. Because the elements that we have to invert, do not come in Hamiltonians derived from the Hamiltonians of the Heisenberg version of the confluent KZ equations, in the reduction process. Unfortunately, the result of [7] can not explain immediately a relation between the (confluent) KZ equations and Schrödinger equations of the quantum Painlevé equations. We study them in the present paper.

We consider following Schrödinger equations for the quantum Painlevé equations:

$$\hbar \frac{\partial}{\partial t} \Phi(x, t) = \widehat{H}_J \left(x, \hbar \frac{\partial}{\partial x}, t \right) \Phi(x, t) \quad (J = \text{II, III, IV, V, VI}), \quad (1.1)$$

where \hbar is a parameter in \mathbb{C} and Hamiltonians \hat{H}_J are obtained from polynomial Hamiltonians of the Painlevé equations by substituting the operators x , $\hbar\partial/\partial x$ into the canonical coordinates q , p (see Definition 2.1). The Schrödinger equations for the quantum Painlevé equations of type III, V, VI have appeared as differential equations satisfied by correlation functions of two dimensional conformal field theory and expected to be differential equations satisfied by instanton partition functions of $SU(2)$ gauge theories [1].

As an example, the quantized Hamiltonian \hat{H}_{II} is

$$\hat{H}_{II} = \frac{1}{2} \left(\hbar \frac{\partial}{\partial x} \right)^2 - \left(x^2 + \frac{t}{2} \right) \hbar \frac{\partial}{\partial x} + ax,$$

where a is a complex parameter. It is easy to see that the different ordering of the operators x and $\partial/\partial x$ can be absorbed in the parameter a . The other quantized Hamiltonians \hat{H}_J also have same property.

Another property of the quantized Hamiltonians is that they act on $\bigoplus_{i=0}^m \mathbb{C}x^i$ if $a = m\hbar$ with a nonnegative integer m . Consequently, we can consider polynomial solutions

$$\Phi(x, t) = \sum_{i=0}^m \varphi_i(t) x^i$$

and the Schrödinger equations for the quantum Painlevé equations (1.1) become linear differential systems for $\varphi_i(t)$ ($i = 0, 1, \dots, m$). We remark that this property is also satisfied by Schrödinger equations for a quantization of other Painlevé systems or isomonodromy deformations, for example, the Garnier system [5].

When $m = 1$ and $J = II$, the linear differential system is

$$\frac{d}{dt}\varphi_0(t) = -\frac{t}{2}\varphi_1(t), \quad \frac{d}{dt}\varphi_1(t) = \varphi_0(t).$$

Namely, $\varphi_1(t)$ satisfies the Airy equation. For the cases of $J = III, IV, V, VI$, $\varphi_1(t)$ satisfy the Bessel equation, the Hermite-Weber equation, the Kummer equation and the Gauss hypergeometric equation, respectively.

In terms of the integral representation, a polynomial solution $\Phi(x, t)$ for $m = 1$ and $J = II$ can be expressed as

$$\Phi(x, t) = \int_{\Gamma} \exp \left(-ut - \frac{2}{3}u^3 \right) (x - u) du$$

for an appropriate cycle Γ . For general $m \in \mathbb{Z}_{\geq 0}$, polynomial solutions are expressed by a generalization of the integral formula above.

Let $\rho_J(u)$ ($J = II, III, IV, V, VI$) be the master functions of the integral formulas of the Airy function, the Bessel function, the Hermite-Weber function, the Kummer function and the Gauss hypergeometric function, respectively (see Definition 2.4).

Theorem 1.1. *For $m \in \mathbb{Z}_{\geq 0}$, the integral formula*

$$\Phi_m^J(x, t) = \int_{\Gamma} \prod_{1 \leq i < j \leq m} (u_i - u_j)^{2\hbar} \prod_{i=1}^m \rho_J(u_i) (x - u_i) du_i \quad (1.2)$$

is a solution of the Schrödinger equations for the quantum Painlevé equation of type J with $a = m\hbar$. Here, Γ is an m -cycle of the homology group determined by the integrand.

These integral formulas have been studied in various fields, such as, two dimensional conformal field theory, random matrix theory (in particular the β -ensembles, for example, see [2] and references therein), the theory of the orthogonal polynomials, and the theory of hypergeometric functions. In particular, if $\hbar = 1$ and $\Gamma = \prod_{i=1}^m \Gamma_1$, where Γ_1 is an appropriate cycle for $\Phi_1^J(x, t)$, the integral formulas $\Phi_m^J(x, t)$ ($m \in \mathbb{Z}_{\geq 0}$) are orthogonal polynomials in x with respect to the weight function $\rho_J(u)$. Because the orthogonal polynomials admit determinant representations (for example, see [17], Chapter II), we have

Corollary 1.2. *If $\hbar = 1$, then determinant formulas*

$$P_m^J(x, t) = \det \left(\begin{pmatrix} \tau_J^{(i+j-2)} \\ (x^{j-1}) \end{pmatrix}_{1 \leq i \leq m, 1 \leq j \leq m+1} \right) = \begin{vmatrix} \tau_J & \tau_J^{(1)} & \cdots & \tau_J^{(m)} \\ \tau_J^{(1)} & \tau_J^{(2)} & \cdots & \tau_J^{(m+1)} \\ \vdots & & & \vdots \\ \tau_J^{(m-1)} & \cdots & \cdots & \tau_J^{(2m-1)} \\ 1 & x & \cdots & x^m \end{vmatrix}, \quad (1.3)$$

where $\tau_J = \tau_J(t) = \int_{\Gamma_1} \rho(u) du$, $\tau_J^{(i)} = \tau_J^{(i)}(t) = \int_{\Gamma_1} u^i \rho(u) du$, are solutions of the Schrödinger equations for the quantum Painlevé equations of type $J = \text{II, III, IV, V, VI}$ with $a = m$.

We note that the coefficients of x^m in $P_m^J(x, t)$

$$(\tau_J)_m = \begin{vmatrix} \tau_J & \tau_J^{(1)} & \cdots & \tau_J^{(m-1)} \\ \tau_J^{(1)} & \tau_J^{(2)} & \cdots & \tau_J^{(m)} \\ \vdots & & & \vdots \\ \tau_J^{(m-1)} & \cdots & \cdots & \tau_J^{(2m-2)} \end{vmatrix}$$

are tau functions of the Painlevé equations P_J ($J = \text{II, III, IV, V, VI}$) (for example, see [4], [10] and references therein). This is because if $\hbar = 1$, then the Schrödinger equations for the quantum Painlevé equations are related to isomonodromy deformations for the Painlevé equations [16].

As mentioned above, the quantum Painlevé equations are related to two dimensional conformal field theory. It is known that the Knizhnik-Zamolodchikov (KZ) equation can be viewed as a quantization of the Schlesinger equations, which is isomonodromy deformation with regular singularities [6], [15]. On the other hand, the sixth Painlevé equation is derived from the Schlesinger equation and the other Painlevé equations are derived from the irregular Schlesinger equations [8]. We give a representation theoretic correspondence between the (confluent) KZ equations and the Schrödinger equations for the quantum Painlevé equations.

Let M_1 , M_2 , and M_3 be Verma modules with highest weights $\gamma_0^{(1)}$, $\gamma_0^{(2)}$, and $\gamma_0^{(3)}$ for \mathfrak{sl}_2 . We consider the KZ equation on the tensor product of three Verma modules $M = M_1 \otimes M_2 \otimes M_3$ with three points $0, t, 1$. Let H_{KZ} the Hamiltonian of the KZ equation.

Theorem 1.3. *For $\gamma_0^{(i)} \notin \mathbb{Z}$ ($i = 1, 2, 3$) and $m \in \mathbb{Z}_{\geq 0}$, the action of H_{KZ} on the space of singular vectors of weight $\sum_{i=1}^3 \gamma_0^{(i)} - 2m$ is equivalent to the action of the quantized Hamiltonian \hat{H}_{VI} on the subspace $\oplus_{i=0}^m \mathbb{C} x^i$ of the polynomial ring $\mathbb{C}[x]$ with $a = m\hbar$.*

In the cases of $J = \text{II, III, IV, V}$, we consider the confluent KZ equations defined in [7].

Theorem 1.4. *For $m \in \mathbb{Z}_{\geq 0}$, the actions of Hamiltonians of the certain confluent KZ equations are equivalent to the actions of the quantized Hamiltonians \hat{H}_J ($J = \text{II}, \text{III}, \text{IV}, \text{V}$) with $a = m\hbar$.*

In these cases, we do not consider the space of singular vectors.

The remainder of this paper is organized as follows. In section 2, we present integral formulas and prove that the integral formulas are solutions to the Schrödinger equations for the quantum Painlevé equations. Moreover, we give determinant formulas for solutions. In section 3, we recall the definition of the (confluent) KZ equations corresponding to the Schrödinger equations for the quantum Painlevé equations and give a representation theoretic correspondence between the (confluent) KZ equations and the Schrödinger equations for the quantum Painlevé equations. We also give a symmetry of the Schrödinger equations for the quantum Painlevé equations with respect to $\hbar \rightarrow -\hbar$ in quantized Hamiltonians.

2. INTEGRAL FORMULA

In this section, we present integral formulas taking values in $\bigoplus_{i=0}^m \mathbb{C}x^i$ ($m \in \mathbb{Z}_{\geq 0}$) and show that they are solutions to the quantum Painlevé equations.

Definition 2.1. *Quantized Hamiltonians \hat{H}_J are defined as*

$$\begin{aligned} t(t-1)\hat{H}_{\text{VI}}\left(x, \hbar\frac{\partial}{\partial x}, a, b, c, d, t\right) &= x(x-1)(x-t)\left(\hbar\frac{\partial}{\partial x}\right)^2 \\ &\quad - ((a+b)(x-1)(x-t) + cx(x-t) + dx(x-1))\hbar\frac{\partial}{\partial x} + (b+c+d+\hbar)a(x-t), \\ t\hat{H}_{\text{V}}\left(x, \hbar\frac{\partial}{\partial x}, a, b, c, t\right) &= x(x-1)\left(\hbar\frac{\partial}{\partial x}\right)^2 + (tx^2 - (b+c+t)x + b)\hbar\frac{\partial}{\partial x} \\ &\quad + a(b+c-a+\hbar+t-tx), \\ \hat{H}_{\text{IV}}\left(x, \hbar\frac{\partial}{\partial x}, a, b, t\right) &= x\left(\hbar\frac{\partial}{\partial x}\right)^2 + (x^2 - tx - b)\hbar\frac{\partial}{\partial x} - ax - (a-2b)t, \\ t\hat{H}_{\text{III}}\left(x, \hbar\frac{\partial}{\partial x}, a, b, t\right) &= x^2\left(\hbar\frac{\partial}{\partial x}\right)^2 - (x^2 + bx + t)\hbar\frac{\partial}{\partial x} + ax, \\ \hat{H}_{\text{II}}\left(x, \hbar\frac{\partial}{\partial x}, a, t\right) &= \frac{1}{2}\left(\hbar\frac{\partial}{\partial x}\right)^2 - \left(x^2 + \frac{t}{2}\right)\hbar\frac{\partial}{\partial x} + ax, \end{aligned}$$

where $a, b, c, d \in \mathbb{C}$.

Remark 2.2. *The spectral problem of the Hamiltonian \hat{H}_{VI} is the Heun equation, which is equivalent to the BC_1 Inozemtsev model. This was known as the Painlevé-Calogero correspondence [9], [18].*

As mentioned in Introduction, the Schrödinger equations for the quantum Painlevé equations have polynomial solutions in x because of the following Proposition.

Proposition 2.3. *The quantized Hamiltonians \hat{H}_J act on $\bigoplus_{i=0}^m \mathbb{C}x^i$ ($m \in \mathbb{Z}_{\geq 0}$) if and only if*

$$\begin{cases} a = m\hbar & \text{or} & b + c + d = (m-1)\hbar & J = \text{VI}, \\ a = m\hbar & & & J = \text{II}, \text{III}, \text{IV}, \text{V}. \end{cases}$$

Proof. It is sufficient to compute the action of \widehat{H}_J on x^m . If $J = \text{II}$, then,

$$\widehat{H}_{\text{II}}x^m = (a - m\hbar)x^{m+1} - \frac{t}{2}m\hbar x^m + \frac{1}{2}m(m-1)\hbar^2x^{m-2}.$$

Hence, \widehat{H}_{II} acts on $\bigoplus_{i=0}^m \mathbb{C}x^i$ if and only if $a = m\hbar$.

For the other cases, we can prove Proposition 2.3 in the same way. \square

From Proposition 2.3, the Schrödinger equations for the quantum Painlevé equations have polynomial solutions $\Phi_m^J(x, t) = \sum_{i=0}^m \varphi_i(t)x^i$ and then become linear differential systems of $\varphi_i(t)$. We show that polynomial solutions $\Phi_m^J(x, t)$ are expressed in terms of integral formulas of hypergeometric type.

Definition 2.4. We define the master functions ρ_J as follows.

$$\begin{aligned}\rho_{\text{VI}}(u, t, a, b, c, d) &= u^{-a-b-1}(1-u)^{-c-1}(t-u)^{-d}, \\ \rho_{\text{V}}(u, t, b, c) &= u^{-b-1}(1-u)^{-c-1}\exp(ut), \\ \rho_{\text{IV}}(u, t, b) &= u^{-b-1}\exp\left(-ut + \frac{u^2}{2}\right), \\ \rho_{\text{III}}(u, t, b) &= u^{-b-1}\exp\left(\frac{t}{u} - u\right), \\ \rho_{\text{II}}(u, t) &= \exp\left(-\left(ut + \frac{2}{3}u^3\right)\right).\end{aligned}$$

Theorem 2.5. For $m \in \mathbb{Z}_{\geq 0}$, the integral formulas

$$\Phi_m^J(x, t, \mathbf{a}, \hbar) = \int_{\Gamma_m^J} \prod_{1 \leq i < j \leq m} (u_i - u_j)^{2\hbar} \prod_{i=1}^m \rho_J(u_i, t, \mathbf{a})(x - u_i) du_i,$$

where Γ_m^J is an m -cycle of the homology group determined by the integrand and

$$\mathbf{a} = \begin{cases} (a, b, c, d) & J = \text{VI}, \\ (b, c) & J = \text{V}, \\ b & J = \text{III, IV}, \end{cases}$$

are solutions to the Schrödinger equations for the quantum Painlevé equations of type $J = \text{II, III, IV, V, VI}$ with $a = m\hbar$ or of type VI with $b + c + d = (m-1)\hbar$.

Let singular loci $U_J \subset \mathbb{C}^m$ ($J = \text{II, III, IV, V, VI}$) be defined as

$$\begin{aligned}U_{\text{VI}} &= \bigcup_{1 \leq i < j \leq m} \{u_i = u_j\} \cup \bigcup_{1 \leq i \leq m} \{u_i = 0\} \cup \bigcup_{1 \leq i \leq m} \{u_i = 1\} \cup \bigcup_{1 \leq i \leq m} \{u_i = t\}, \\ U_{\text{V}} &= \bigcup_{1 \leq i < j \leq m} \{u_i = u_j\} \cup \bigcup_{1 \leq i \leq m} \{u_i = 0\} \cup \bigcup_{1 \leq i \leq m} \{u_i = 1\}, \\ U_{\text{IV}} = U_{\text{III}} &= \bigcup_{1 \leq i < j \leq m} \{u_i = u_j\} \cup \bigcup_{1 \leq i \leq m} \{u_i = 0\}, \\ U_{\text{II}} &= \bigcup_{1 \leq i < j \leq m} \{u_i = u_j\}.\end{aligned}$$

For a rational function $\varphi(u_1, \dots, u_m)$ holomorphic outside U_J , denote by $\langle \varphi(u_1, \dots, u_m) \rangle_m$ an integral formula

$$\int_{\Gamma_m^J} \prod_{1 \leq i < j \leq m} (u_i - u_j)^{2\hbar} \prod_{i=1}^m \rho_J(u_i, \mathbf{a}) \varphi(u_1, \dots, u_m) du_i.$$

Let ∇_i ($i = 1, \dots, m$) be the differential defined by

$$\nabla_i = \frac{\partial}{\partial u_i} + \frac{\partial}{\partial u_i} \left(\log \left(\prod_{1 \leq i < j \leq m} (u_i - u_j)^{2\hbar} \prod_{i=1}^m \rho_J(u_i, t, \mathbf{a}) \right) \right).$$

By definition, it holds

$$\langle \nabla_i (\varphi(u_1, \dots, u_m)) \rangle = \int_{\Gamma_m^J} \prod_{j=1}^m du_j \frac{\partial}{\partial u_i} \left(\prod_{1 \leq i < j \leq m} (u_i - u_j)^{2\hbar} \prod_{i=1}^m \rho_J(u_i, \mathbf{a}) \varphi(u_1, \dots, u_m) \right) = 0.$$

Note that the polynomial $\prod_{i=1}^m (x - u_i)$ in the integral formula is symmetrical with respect to the integral variables u_1, \dots, u_m . It is convenient to use the symmetrization for computations of the integral formula. Let $\text{Sym}[\varphi(u_1, \dots, u_m)]$ be the symmetrization given by

$$\text{Sym}[\varphi(u_1, \dots, u_m)] = \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} \varphi(u_{\sigma(1)}, \dots, u_{\sigma(m)}),$$

where \mathfrak{S}_m is the symmetric group of degree m .

A proof of Theorem 2.5 for the case of $J = \text{VI}$. Let $\Phi_m^{\text{VI}}(x, t, a, b, c, d, \hbar)$ be written as

$$\Phi_m^{\text{VI}}(x, t, a, b, c, d, \hbar) = \sum_{k=0}^m (-1)^{k+1} \binom{m}{k} \left\langle \text{Sym} \left[\prod_{l=1}^{m-k} u_l \right] \right\rangle_m x^k,$$

where $\binom{m}{k}$ is the binomial coefficient. Denote $\text{Sym} \left[\prod_{l=1}^{m-k} u_l \right]$ by φ_k . For $k = 0, 1, \dots, m$, we have

$$\frac{\partial}{\partial t} \langle \varphi_k \rangle_m = (k-m)d \left\langle \text{Sym} \left[\left(\frac{t}{t - u_{m-k}} - 1 \right) \prod_{l=1}^{m-k-1} u_l \right] \right\rangle_m - kd \left\langle \text{Sym} \left[\frac{1}{t - u_{m-k+1}} \prod_{l=1}^{m-k} u_l \right] \right\rangle_m.$$

Applying Lemma 2.6 below, we obtain

$$\begin{aligned} t(t-1) \frac{\partial}{\partial t} \langle \varphi_k \rangle_m &= (m-k)t(a+b-k\hbar) \langle \varphi_{k+1} \rangle_m \\ &\quad - \{(m-k)t(a+b+c+d-(m+k-1)\hbar) \\ &\quad - k(a+b+d(1-t)-(k-1)\hbar)\} \langle \varphi_k \rangle_m \\ &\quad - k(a+b+c+d-(m+k-2)\hbar) \langle \varphi_{k-1} \rangle_m. \end{aligned} \tag{2.1}$$

On the other hand, the action of the Hamiltonian \widehat{H}_{VI} on $\Phi_m^{\text{VI}}(x, t, a, b, c, d, \hbar)$ is easily calculated and we see that the coefficient of x^k of

$$t(t-1) \frac{\widehat{H}_{\text{VI}}}{\hbar} \Phi_m^{\text{VI}}(x, t, a, b, c, d, \hbar)$$

times $(-1)^{k+1} \binom{m}{k}^{-1}$ coincides with the right hand side of (2.1), which finishes the proof. \square

Lemma 2.6. *For, $k = 1, \dots, m$, we have*

$$t(t-1) \left\langle \text{Sym} \left[\frac{d}{t - u_{m-k}} \prod_{l=1}^{m-k-1} u_l \right] \right\rangle_m = - (a + b + d(1-t) - k\hbar) \langle \varphi_{k+1} \rangle_m \quad (2.2)$$

$$+ (a + b + c + d - (m+k-1)\hbar) \langle \varphi_k \rangle_m.$$

Proof. In order to prove (2.2), we compute

$$X = \left\langle \text{Sym} \left[\nabla_i \left(u_1(1-u_1) \prod_{l=2}^{m-k} u_l \right) \right] \right\rangle = 0$$

as follows. From the definition, we get

$$X = - (a + b + d(1-t)) \langle \varphi_{k+1} \rangle_m + (a + b + c + d) \langle \varphi_k \rangle_m - t(t-1) \left\langle \text{Sym} \left[\frac{d}{t - u_{m-k}} \prod_{l=1}^{m-k-1} u_l \right] \right\rangle_m$$

$$+ 2\hbar \left\langle \text{Sym} \left[\sum_{l=2}^m \frac{u_1(1-u_1)}{u_1 - u_l} u_2 \cdots u_{m-k} \right] \right\rangle_m.$$

Since the symmetrization is invariant under the action of \mathfrak{S}_m on the integral variables u_1, \dots, u_m , we have

$$2 \left\langle \text{Sym} \left[\sum_{l=2}^m \frac{u_1(1-u_1)}{u_1 - u_l} u_2 \cdots u_{m-k} \right] \right\rangle_m = \left\langle \text{Sym} \left[\sum_{l=2}^m \frac{u_1(1-u_1)}{u_1 - u_l} u_2 \cdots u_{m-k} \right] \right\rangle_m$$

$$- \left\langle \text{Sym} \left[\sum_{l=2}^{m-k} \frac{u_l(1-u_l)}{u_1 - u_l} u_1 u_2 \cdots u_{l-1} u_{l+1} \cdots u_{m-k} \right] \right\rangle_m$$

$$- \left\langle \text{Sym} \left[\sum_{l=m-k+1}^m \frac{u_l(1-u_l)}{u_1 - u_l} u_2 \cdots u_{m-k} \right] \right\rangle_m$$

$$= k \langle \varphi_{k+1} \rangle_m - (m+k-1) \langle \varphi_k \rangle_m.$$

Therefore, we arrive at (2.2). \square

For the other cases, we can verify Theorem 2.5 in the similar way. It also follows from Theorem 3.4, Theorem 3.6, Theorem 3.8, Theorem 3.10 in section 3, and Proposition 4.2 in [7]. We give an another proof of Theorem 2.5 for the case of $J = V$ in section 3.

As mentioned in Introduction, if $\hbar = 1$ and $\Gamma_m^J = \prod_{i=1}^m \Gamma_1^J$, where Γ_1^J is an appropriate cycle for $\Phi_1^J(x, t)$, then the integral formulas $\Phi_m^J(x, t)$ ($m \in \mathbb{Z}_{\geq 0}$) are expressed by the determinant formulas $P_m^J(x, t)$ (1.3) and $P_m^J(x, t)$ are orthogonal polynomials in x with respect to the weight functions $\rho_J(x)$. Namely, it holds that

$$\int_{\Gamma_1^J} P_m^J(x, t) P_n^J(x, t) \rho_J(x) dx = 0.$$

From Theorem 2.5, we have

Corollary 2.7. *If $\hbar = 1$, then the determinant formulas $P_m^J(x, t)$ ($J = \text{II, III, IV, V, VI}$) are solutions to the Schrödinger equations for the quantum Painlevé equations of type II, III, IV, V, VI with $a = m$ or of type VI with $b + c + d = m - 1$.*

3. RELATION TO THE KZ EQUATION

In this section, we give a representation theoretic correspondence between the Schrödinger equations for the sixth quantum Painlevé equation and the KZ equation, and between the Schrödinger equations for the other quantum Painlevé equations and the confluent KZ equations, which were defined directly in [7] and can be derived from the irregular conformal field theory [13]. In what follows, we only give necessary facts on confluent KZ equations in this paper. See [7] for the detail.

Let us recall the definition of confluent Verma modules in [7]. Set $\mathfrak{g} = \mathfrak{sl}_2$ and $\mathfrak{g}[z] = \mathfrak{g} \otimes \mathbb{C}[z]$. Denote by e, f, h the standard basis of \mathfrak{g} . For non-negative integer r , denote by $\mathfrak{g}_{(r)}$ and $\mathfrak{g}'_{(r)}$ the truncated Lie algebras $\mathfrak{g}[z]/z^{r+1}\mathfrak{g}[z]$ and $\mathfrak{g}'_{(r)} = z\mathfrak{g}[z]/z^{r+1}\mathfrak{g}[z]$, respectively. For an $(r+1)$ -tuple parameters $\gamma = (\gamma_0, \dots, \gamma_{r-1}, \gamma_r) \in \mathbb{C}^r \times \mathbb{C}^\times$, a confluent Verma module $M(\gamma)$ of Poincaré rank r is a cyclic $\mathfrak{g}_{(r)}$ -module generated by $\mathbf{1}_\gamma$ such that

$$(e \otimes z^p)\mathbf{1}_\gamma = 0, \quad (h \otimes z^p)\mathbf{1}_\gamma = \gamma_p \mathbf{1}_\gamma \quad (0 \leq p \leq r).$$

For the Lie subalgebra $\mathfrak{g}'_{(r)} = z\mathfrak{g}[z]/z^{r+1}\mathfrak{g}[z]$, a confluent Verma module $M'(\gamma)$ of Poincaré rank r with parameters $\gamma = (\gamma_1, \dots, \gamma_r) \in \mathbb{C}^{r-1} \times \mathbb{C}^\times$ is a cyclic $\mathfrak{g}'_{(r)} \oplus \mathbb{C}(h \otimes z^0)$ -module generated by $\mathbf{1}_\gamma$ such that

$$(e \otimes z^p)\mathbf{1}_\gamma = 0, \quad (h \otimes z^p)\mathbf{1}_\gamma = \gamma_p \mathbf{1}_\gamma \quad (1 \leq p \leq r), \quad (h \otimes z^0)\mathbf{1}_\gamma = 0,$$

and $e \otimes z^r$ and $f \otimes z^r$ act as zero operators on $M'(\gamma)$.

Let differential operators D_k ($0 \leq k \leq r-1$) be defined as

$$D_k = \sum_{p=1}^{r-k} p\gamma_{k+p} \frac{\partial}{\partial \gamma_p}$$

acting on $M(\gamma)$ as

$$D_k(x \otimes z^p) = p(x \otimes z^{p+k}) \quad (x \in \mathfrak{g}, 0 \leq p \leq r), \quad D_k(\mathbf{1}_\gamma) = 0.$$

Here we regard $x \otimes z^p$ as an operator on $M(\gamma)$.

Example 3.1. *For the $\mathfrak{g}'_{(2)}$ case, $M'(\gamma_1, \gamma_2)$ is realized on the polynomial ring $\mathbb{C}[x]$. The action of $x \otimes z^p$ ($x \in \mathfrak{g}$, $p = 1, 2$) are following.*

$$e \otimes z = \gamma_2^{\frac{1}{2}} \frac{\partial}{\partial x}, \quad e \otimes z^2 = 0, \quad f \otimes z = \gamma_2^{\frac{1}{2}} x, \quad f \otimes z^2 = 0, \quad h \otimes z = \gamma_1, \quad h \otimes z^2 = \gamma_2.$$

Let z_1, \dots, z_n be distinct points in \mathbb{C} and let $r_1, \dots, r_n, r_\infty$ be non-negative integers. Set $\mathfrak{a} = (\oplus_{i=1}^n \mathfrak{g}^{(i)}) \oplus \mathfrak{g}^{(\infty)}$, where $\mathfrak{g}^{(i)} = \mathfrak{g}_{(r_i)}$ ($i = 1, \dots, n$) and $\mathfrak{g}^{(\infty)} = \mathfrak{g}'_{(r_\infty)}$. We consider a family of \mathfrak{a} -modules

$$M(\gamma) = M^{(1)} \otimes \dots \otimes M^{(n)} \otimes M^{(\infty)},$$

parametrized by $\gamma = (\gamma^{(1)}, \dots, \gamma^{(n)}, \gamma^{(\infty)})$, where

$$\begin{aligned} M^{(i)} &= M(\gamma^{(i)}), \quad \gamma^{(i)} = (\gamma_0^{(i)}, \dots, \gamma_{r_i}^{(i)}) \in \mathbb{C}^{r_i} \times \mathbb{C}^\times, \\ M^{(\infty)} &= M'(\gamma^{(\infty)}), \quad \gamma^{(\infty)} = (\gamma_1^{(\infty)}, \dots, \gamma_{r_\infty}^{(\infty)}) \in \mathbb{C}^{r_\infty-1} \times \mathbb{C}^\times. \end{aligned}$$

Set $\mathbf{1}_\gamma = \mathbf{1}_{\gamma^{(1)}} \otimes \dots \otimes \mathbf{1}_{\gamma^{(n)}} \otimes \mathbf{1}_{\gamma^{(\infty)}}$.

The confluent KZ equations defined in [7] are differential systems for unknown functions $\Phi(z, \gamma)$ taking values in $M(\gamma)$ with respect to the following differential operators

$$\begin{aligned} \frac{\partial}{\partial z_i} \quad (i = 1, \dots, n), \\ D_k^{(i)} \quad (i = 1, \dots, n, k = 0, \dots, r_i - 1), \\ D_k^{(\infty)} \quad (k = 1, \dots, r_\infty - 1). \end{aligned}$$

If $r_i = 0$ ($1 \leq i \leq n$) and $r_\infty = 0$, then the confluent KZ equations are equal to the usual KZ equations.

It was shown in [7] that the confluent KZ equations have integral formulas of confluent hypergeometric type for solutions.

3.1. The case of J = VI. Let $n = 3$, $r_i = 0$, $z_1 = 0$, $z_2 = t$, $z_3 = 1$ and $\gamma_0^{(i)} \notin \mathbb{Z}$ ($1 \leq i \leq 3$). Then $M = M(\gamma_0^{(1)}) \otimes M(\gamma_0^{(2)}) \otimes M(\gamma_0^{(3)})$ and the KZ equation for an unknown function $\Phi(t)$ taking values in M is

$$\kappa \frac{\partial \Psi(t)}{\partial t} = \left(\frac{\Omega^{(1,2)}}{t} + \frac{\Omega^{(2,3)}}{t-1} \right) \Psi(t). \quad (3.1)$$

Here κ is a complex parameter and $\Omega^{(i,j)}$ are the Casimir operators:

$$\Omega^{(1,2)} = e^{(1)} f^{(2)} + f^{(1)} e^{(2)} + \frac{1}{2} h^{(1)} h^{(2)}, \quad \Omega^{(2,3)} = e^{(2)} f^{(3)} + f^{(2)} e^{(3)} + \frac{1}{2} h^{(2)} h^{(3)},$$

where $x^{(i)} : M \rightarrow M$ is the linear operator acting as x on i th tensor factor and as identities on the others, and here and after, we abbreviate $x \otimes z^0$ to x , for $x = e, f, h$.

Let W_m ($m \in \mathbb{Z}_{\geq 0}$) be the space of singular vectors of the weight $\sum_{i=1}^3 \gamma_0^{(i)} - 2m$ in M , namely,

$$W_m = \left\{ v \in M \left| \sum_{i=1}^3 e^{(i)}(v) = 0, \sum_{i=1}^3 h^{(i)}(v) = \left(\sum_{i=1}^3 \gamma_0^{(i)} - 2m \right) v \right. \right\}.$$

If $\gamma_0^{(i)} \notin \mathbb{Z}$, then the dimension of W_m is known to be $m+1$ (for example, see Proposition 4.1.1 in [3]).

In order to write down a basis of W_m , we take the differential realizations $\mathbb{C}[x_i]$ ($1 \leq i \leq 3$) of \mathfrak{g} , that is, the basis e, f, h of \mathfrak{g} act on $\mathbb{C}[x_i]$ as follows:

$$e = \partial_i, \quad h = -2x_i \partial_i + \gamma_0^{(i)}, \quad f = -x_i^2 \partial_i + \gamma_0^{(i)} x_i,$$

where $\partial_i = \partial/\partial x_i$. Note that if $\gamma_0^{(i)} \notin \mathbb{Z}$, then $\mathbb{C}[x_i]$ are isomorphic to Verma modules $M(\gamma_0^{(i)})$. We set $M(\gamma_0^{(i)}) = \mathbb{C}[x_i]$.

The space of singular vectors W_m can be written by

$$W_m = \bigoplus_{i=0}^m \mathbb{C}(x_1 - x_2)^i (x_1 - x_3)^{m-i}. \quad (3.2)$$

We denote by H_{KZ} the Hamiltonian $\Omega^{(1,2)}/t + \Omega^{(2,3)}/(t-1)$. Let $\tilde{H}_{\text{KZ}}(m)$ be defined as

$$\tilde{H}_{\text{KZ}}(m) = \hbar^2 \left(H_{\text{KZ}} - \frac{\lambda_1 \lambda_2}{t} - \frac{\lambda_2 (\lambda_3 - m)}{t-1} \right).$$

We define linear isomorphisms $T_m : W_m \rightarrow \bigoplus_{i=0}^m \mathbb{C}x^i$ ($m \in \mathbb{Z}_{\geq 0}$) as

$$T_m \left((x_1 - x_2)^i (x_1 - x_3)^{m-i} \right) = x^i \quad (0 \leq i \leq m).$$

Theorem 3.2. For $\gamma_0^{(i)} \notin \mathbb{Z}$ ($1 \leq i \leq 3$) and $m \in \mathbb{Z}_{\geq 0}$, the action of H_{KZ} on the space of singular vectors of weight $\sum_{i=1}^3 \gamma_0^{(i)} - 2m$ is equivalent to the action of the quantized Hamiltonian \hat{H}_{VI} on the subspace $\bigoplus_{i=0}^m \mathbb{C}x^i$ with $a = m\hbar$ or $b + c + d = (m-1)\hbar$. Namely, we have

$$T_m \circ \tilde{H}_{\text{KZ}}(m) = \left(\hat{H}_{\text{VI}} \left(x, \hbar \frac{\partial}{\partial x}, a, b, c, d, t \right) + \frac{a(b+c+d+\hbar)}{t-1} \right) \circ T_m \quad (3.3)$$

as linear maps from W_m to $\bigoplus_{i=0}^m \mathbb{C}x^i$ with

$$\gamma_0^{(1)} = m - 1 - \frac{c}{\hbar}, \quad (3.4)$$

$$\gamma_0^{(2)} = \frac{1}{\hbar} (a + b + c + d + (1-m)\hbar), \quad \gamma_0^{(3)} = m - 1 - \frac{a+b}{\hbar}, \quad (3.5)$$

and

$$a = m\hbar \quad \text{or} \quad b + c + d = (m-1)\hbar. \quad (3.6)$$

Proof. It follows from direct computations. \square

We note that the condition $\gamma_0^{(i)} \notin \mathbb{Z}$ ensures $M(\gamma_0^{(i)}) = \mathbb{C}[x_i]$, and for any $\gamma_0^{(i)} \in \mathbb{C}$ ($1 \leq i \leq 3$), it holds (3.3) as linear maps from $\bigoplus_{i=0}^m \mathbb{C}(x_1 - x_2)^i (x_1 - x_3)^{m-i}$ to $\bigoplus_{i=0}^m \mathbb{C}x^i$ with the conditions (3.4), (3.5), and (3.6).

From Theorem 3.2, we obtain a solution $\Psi(t)$ to the KZ equation (3.1) from the integral formula $\Phi_m^{\text{VI}}(x, t)$ for a solution to the Schrödinger equation for the sixth quantum Painlevé equation, that is,

$$\Psi(t) = t^{(\gamma_0^{(1)}\gamma_0^{(2)})/\kappa} (t-1)^{(\gamma_0^{(2)}\gamma_0^{(3)})/\kappa} T_m^{-1}(\Phi_m^{\text{VI}}(x, t))$$

obtained by replacing $\hbar, a+b, c, d$ with

$$\frac{1}{\kappa}, \quad a+b = \hbar(m-1-\gamma_0^{(3)}), \quad c = \hbar(m-1-\gamma_0^{(1)}), \quad d = \hbar(\gamma_0^{(1)} + \gamma_0^{(2)} + \gamma_0^{(3)} + 1 - m),$$

respectively. Conversely, from Theorem 3.2, we obtain

$$\left(\hbar^2 \kappa \frac{\partial}{\partial t} - \hat{H}_{\text{VI}} \right) T_m \left(\tilde{\Psi}(t) \right) = 0,$$

where $\tilde{\Psi}(t) = t^{-(\gamma_0^{(1)}\gamma_0^{(2)})/\kappa}(t-1)^{-(\gamma_0^{(2)}\gamma_0^{(3)})/\kappa}\Psi(t)$. If $\kappa = 1/\hbar$, then $T_m(\tilde{\Psi}(t)) = \Phi_m^{\text{VI}}(x, t)$. Taking $\kappa = -1/\hbar$, we have

$$\left(\hbar\frac{\partial}{\partial t} + \widehat{H}_{\text{IV}}\right)\Phi_m^{\text{VI}}(x, t, -a, -b, -c, -d, -\hbar) = 0.$$

This implies the following symmetry.

Proposition 3.3. *If a function $\Phi^{\text{VI}}(x, t, a, b, c, d, \hbar)$ is a solution to the Schrödinger equation for the quantum sixth Painlevé equation, then the function $\Psi^{\text{VI}}(x, t, a, b, c, d, \hbar)$ defined as*

$$\Psi^{\text{VI}}(x, t, a, b, c, d, \hbar) = \Phi^{\text{VI}}(x, t, -a, -b, -c, -d, \hbar) \quad (3.7)$$

is a solution to

$$\hbar\frac{\partial}{\partial t}\Psi^{\text{VI}}(x, t) = \widehat{H}'_{\text{VI}}\Psi^{\text{VI}}(x, t),$$

where \widehat{H}'_{VI} are obtained by replacing \hbar with $-\hbar$ in \widehat{H}_{VI} .

Proof. It follows from direct computations. □

3.2. The case of $J = V$. Let $n = 2$, $r_1 = r_2 = 0$, $r_\infty = 1$, $z_1 = 0$, and $z_2 = 1$. Then $M = M(\gamma_0^{(1)}) \otimes M(\gamma_0^{(2)}) \otimes M'(\gamma_1^{(\infty)})$ and the confluent KZ equation for an unknown function $\Phi(\gamma_1^{(\infty)})$ taking values in M is

$$\kappa\gamma_1^{(\infty)}\frac{\partial}{\partial\gamma_1^{(\infty)}}\Psi(\gamma_1^{(\infty)}) = \left(G_0^{(\infty)} - \frac{1}{4}h_0(h_0 + 2)\right)\Psi(\gamma_1^{(\infty)}), \quad (3.8)$$

where $\kappa \in \mathbb{C}$, $G_0^{(\infty)} = G_{-1}^{(2)} + G_0^{(1)} + G_0^{(2)}$,

$$\begin{aligned} G_{-1}^{(2)} &= -\frac{1}{2}h^{(2)}\gamma_1^{(\infty)} + e^{(1)}f^{(2)} + f^{(1)}e^{(2)} + \frac{1}{2}h^{(1)}h^{(2)}, \\ G_0^{(i)} &= \frac{1}{2}\left(e^{(i)}f^{(i)} + f^{(i)}e^{(i)} + \frac{1}{2}h^{(i)}h^{(i)}\right) \quad (i = 1, 2), \\ h_0 &= h^{(1)} + h^{(2)}. \end{aligned}$$

Let M_m ($m \in \mathbb{Z}_{\geq 0}$) be the weight space of M with the weight $\gamma_0^{(1)} + \gamma_0^{(2)} - 2m$, namely,

$$\begin{aligned} M_m &= \left\{v \in M \mid h_0(v) = (\gamma_0^{(1)} + \gamma_0^{(2)} - 2m)v\right\} \\ &= \bigoplus_{i=0}^m \mathbb{C}(f^{(1)})^i(f^{(2)})^{m-i}\mathbf{1}_\gamma. \end{aligned}$$

An integral formula taking values in M_m for solutions to the confluent KZ equation (3.8) is

$$\begin{aligned} &\int_\Gamma \prod_{i=1}^m du_i \exp\left(-\frac{1}{2\kappa}\gamma_0^{(2)}\gamma_1^{(\infty)}\right) \prod_{1 \leq i < j \leq m} (u_i - u_j)^{2/\kappa} \prod_{i=1}^m u_i^{-\gamma_0^{(1)}/\kappa} (1 - u_i)^{-\gamma_0^{(2)}/\kappa} \\ &\quad \times \exp\left(\frac{1}{\kappa} \sum_{i=1}^m u_i \gamma_1^{(\infty)}\right) \prod_{i=1}^m \left(\frac{f^{(1)}}{u_i} + \frac{f^{(2)}}{u_i - 1}\right) \mathbf{1}_\gamma, \end{aligned} \quad (3.9)$$

where Γ is an appropriate cycle (see Proposition 4.2 in [7]).

We define linear isomorphisms $T_m : M_m \rightarrow \bigoplus_{i=0}^m \mathbb{C}x^i$ ($m \in \mathbb{Z}_{\geq 0}$) as

$$T_m \left((f^{(1)})^i (f^{(1)} + f^{(2)})^{m-i} \mathbf{1}_\gamma \right) = x^i \quad (0 \leq i \leq m).$$

Theorem 3.4. *For $m \in \mathbb{Z}_{\geq 0}$, we have*

$$T_m \circ \left(G_0^{(\infty)} - \frac{1}{4}h_0(h_0 + 2) + \frac{1}{2}\gamma_0^{(2)}\gamma_1^{(\infty)} \right) = \left(\frac{t}{\hbar^2} \widehat{H}_V \left(x, \hbar \frac{\partial}{\partial x}, m\hbar, b, c, t \right) \right) \circ T_m$$

as linear maps from M_m to $\bigoplus_{i=0}^m \mathbb{C}x^i$ with

$$\gamma_0^{(1)} = \frac{b}{\hbar}, \quad \gamma_0^{(2)} = \frac{c}{\hbar}, \quad \gamma_1^{(\infty)} = \frac{t}{\hbar}. \quad (3.10)$$

Proof. It follows from direct computations. \square

A proof of Theorem 2.5 for the case of $J = V$. Substitute (3.10) into (3.9). Then we have

$$\left(\kappa t \frac{\partial}{\partial t} - G_0^{(\infty)} + \frac{1}{4}h_0(h_0 + 2) - \frac{ct}{2\hbar^2} \right) \Phi(t) = 0,$$

where

$$\begin{aligned} \Phi(t) = & \int_{\Gamma} \prod_{i=1}^m du_i \prod_{1 \leq i < j \leq m} (u_i - u_j)^{2/\kappa} \prod_{i=1}^m u_i^{-b/(\hbar\kappa)-1} (1 - u_i)^{-c/(\hbar\kappa)-1} \\ & \times \exp \left(\frac{1}{\hbar\kappa} \sum_{i=1}^m u_i t \right) \prod_{i=1}^m (f^{(1)} - (f^{(1)} + f^{(2)})u_i) \mathbf{1}_\gamma. \end{aligned} \quad (3.11)$$

From Theorem 3.4, we obtain

$$\left(\kappa t \frac{\partial}{\partial t} - \frac{t}{\hbar^2} \widehat{H}_V \right) T_m(\Phi(t)) = 0. \quad (3.12)$$

If $\kappa = 1/\hbar$, then (3.12) becomes

$$\left(\hbar \frac{\partial}{\partial t} - \widehat{H}_V \left(x, \hbar \frac{\partial}{\partial x}, m\hbar, b, c, t \right) \right) \Phi_m^V(x, t, b, c, \hbar) = 0.$$

\square

From (3.12), an equation

$$\hbar^2 \kappa \frac{\partial}{\partial t} \Phi(x, t) = \widehat{H}_V \left(x, \hbar \frac{\partial}{\partial x}, m\hbar, b, c, t \right) \Phi(x, t)$$

also has integral formulas for solutions. Especially, taking $\kappa = -1/\hbar$, we have

$$\left(\hbar \frac{\partial}{\partial t} + \widehat{H}_V \left(x, \hbar \frac{\partial}{\partial x}, m\hbar, b, c, t \right) \right) \Phi_m^V(x, -t, -b, -c, -\hbar) = 0$$

from (3.11). This implies the following symmetry.

Proposition 3.5. *If a function $\Phi^V(x, t, a, b, c, \hbar)$ is a solution to the Schrödinger equation for the quantum fifth Painlevé equation, then the function $\Psi^V(x, t, a, b, c, \hbar)$ defined as*

$$\Psi^V(x, t, a, b, c, \hbar) = \Phi^V(x, -t, -a, -b, -c, \hbar) \quad (3.13)$$

is a solution to

$$\hbar \frac{\partial}{\partial t} \Psi^V(x, t) = \widehat{H}'_V \Psi^V(x, t),$$

where \widehat{H}'_V are obtained by replacing \hbar with $-\hbar$ in \widehat{H}_V .

Proof. It follows from direct computations. \square

3.3. The case of $J = IV$. Let $n = 1$, $r_1 = 0$, $r_\infty = 2$, and $z_1 = 0$. Then $M = M(\gamma_0^{(1)}) \otimes M'(\gamma_1^{(\infty)}, \gamma_2^{(\infty)})$ and the confluent KZ equation for an unknown function $\Psi(\gamma_1^{(\infty)})$ taking values in M is

$$\kappa \gamma_2^{(\infty)} \frac{\partial}{\partial \gamma_1^{(\infty)}} \Psi(\gamma_1^{(\infty)}) = \left(G_1^{(\infty)} + \frac{1}{2} h_0 \gamma_1^{(\infty)} \right) \Psi(\gamma_1^{(\infty)}), \quad (3.14)$$

where $\kappa \in \mathbb{C}$,

$$\begin{aligned} G_1^{(\infty)} &= -e^{(1)}(f \otimes z)^{(\infty)} + f^{(1)}(e \otimes z)^{(\infty)} + \frac{1}{2} h^{(1)} \gamma_1^{(\infty)}, \\ h_0 &= h^{(1)} + h^{(\infty)}. \end{aligned}$$

Here $x^{(\infty)}$ for $x \in \mathfrak{g}'_{(2)} \oplus \mathbb{C}(h \otimes 1)$ stands for the linear operator acting as x on $M'(\gamma_1^{(\infty)}, \gamma_2^{(\infty)})$ and as identity on the other.

Let M_m ($m \in \mathbb{Z}_{\geq 0}$) be the weight space of M with the weight $\gamma_0^{(1)} - 2m$, namely,

$$\begin{aligned} M_m &= \left\{ v \in M \mid h_0(v) = (\gamma_0^{(1)} - 2m) v \right\} \\ &= \bigoplus_{i=0}^m \mathbb{C} (f^{(1)})^i ((f \otimes z)^{(\infty)})^{m-i} \mathbf{1}_\gamma. \end{aligned}$$

An integral formula taking values in M_m for solutions to the confluent KZ equation (3.14) is

$$\begin{aligned} & \int_{\Gamma} \prod_{i=1}^m du_i \prod_{1 \leq i < j \leq m} (u_i - u_j)^{2/\kappa} \prod_{i=1}^m u_i^{-\gamma_0^{(1)}/\kappa} \\ & \times \exp \left(\frac{1}{\kappa} \sum_{i=1}^m \left(u_i \gamma_1^{(\infty)} + \frac{u_i^2}{2} \gamma_2^{(\infty)} \right) \right) \prod_{i=1}^m \left(\frac{f^{(1)}}{u_i} - (f \otimes z)^{(\infty)} \right) \mathbf{1}_\gamma, \end{aligned} \quad (3.15)$$

where Γ is an appropriate cycle (see Proposition 4.2 in [7]).

We define linear isomorphisms $T_m : M_m \rightarrow \bigoplus_{i=0}^m \mathbb{C} x^i$ ($m \in \mathbb{Z}_{\geq 0}$) as

$$T_m \left((f^{(1)})^i ((f \otimes z)^{(\infty)})^{m-i} \mathbf{1}_\gamma \right) = x^i \quad (0 \leq i \leq m).$$

Theorem 3.6. For $m \in \mathbb{Z}_{\geq 0}$, we have

$$T_m \circ \left(G_1^{(\infty)} + \frac{1}{2} h_0 \gamma_1^{(\infty)} \right) = \frac{1}{\hbar^2} \widehat{H}_{IV} \left(x, \hbar \frac{\partial}{\partial x}, m\hbar, b, t \right) \circ T_m$$

as linear maps from M_m to $\bigoplus_{i=0}^m \mathbb{C} x^i$ with

$$\gamma_0^{(1)} = \frac{b}{\hbar}, \quad \gamma_1^{(\infty)} = -\frac{t}{\hbar}, \quad \gamma_2^{(\infty)} = \frac{1}{\hbar}.$$

Proof. It follows from direct computations. \square

As in the cases of $J = V, VI$, Theorem 3.6 implies the following symmetry.

Proposition 3.7. *If a function $\Phi^{IV}(x, t, a, b, \hbar)$ is a solution to the Schrödinger equation for the quantum fourth Painlevé equation, then the function $\Psi^{IV}(x, t, a, b, \hbar)$ defined as*

$$\Psi^{IV}(x, t, a, b, \hbar) = \Phi^{IV}(\sqrt{-1}x, \sqrt{-1}t, -a, -b, \hbar) \quad (3.16)$$

is a solution to

$$\hbar \frac{\partial}{\partial t} \Psi^{IV}(x, t) = \widehat{H}'_{IV} \Psi^{IV}(x, t),$$

where \widehat{H}'_{IV} are obtained by replacing \hbar with $-\hbar$ in \widehat{H}_{IV} .

Proof. It follows from direct computations. \square

3.4. The case of $J = III$. Let $n = 1$, $r_1 = 2$, $r_\infty = 1$, and $z_1 = 0$. Then $M = M(\gamma_0^{(1)}, \gamma_1^{(1)}) \otimes M'(\gamma_1^{(\infty)})$ and the confluent KZ equation for an unknown function $\Psi(\gamma_1^{(\infty)})$ taking values in M is

$$\kappa \gamma_1^{(\infty)} \frac{\partial}{\partial \gamma_1^{(\infty)}} \Psi(\gamma_1^{(\infty)}) = \left(G_0^{(\infty)} - \frac{1}{4} h_0(h_0 + 2) \right) \Psi(\gamma_1^{(\infty)}), \quad (3.17)$$

where $\kappa \in \mathbb{C}$,

$$G_0^{(\infty)} = \frac{1}{2} \left(e^{(1)} f^{(1)} + f^{(1)} e^{(1)} + \frac{1}{2} h^{(1)} h^{(1)} \right) - \frac{1}{2} (h \otimes z)^{(1)} \gamma_1^{(\infty)},$$

$$h_0 = h^{(1)} + h^{(\infty)}.$$

Let M_m ($m \in \mathbb{Z}_{\geq 0}$) be the weight space of M with the weight $\gamma_0^{(1)} - 2m$, namely,

$$M_m = \left\{ v \in M \mid h_0(v) = \left(\gamma_0^{(1)} - 2m \right) v \right\}$$

$$= \bigoplus_{i=0}^m \mathbb{C} (f^{(1)})^i ((f \otimes z)^{(1)})^{m-i} \mathbf{1}_\gamma.$$

An integral formula taking values in M_m for a solution to the confluent KZ equation (3.17) is

$$\int_\Gamma \prod_{i=1}^m du_i \prod_{1 \leq i < j \leq m} (u_i - u_j)^{2/\kappa} \prod_{i=1}^m u_i^{-\gamma_0^{(1)}/\kappa} \quad (3.18)$$

$$\times \exp \left(\frac{1}{\kappa} \sum_{i=1}^m \left(\frac{\gamma_1^{(1)}}{u_i} + u_i \gamma_1^{(\infty)} \right) \right) \prod_{i=1}^m \left(\frac{f^{(1)}}{u_i} + \frac{(f \otimes z)^{(1)}}{u_i^2} \right) \mathbf{1}_\gamma,$$

where Γ is an appropriate cycle (see Proposition 4.2 in [7]).

We define linear isomorphisms $T_m : M_m \rightarrow \bigoplus_{i=0}^m \mathbb{C} x^i$ ($m \in \mathbb{Z}_{\geq 0}$) as

$$T_m \left((f^{(1)})^i ((f \otimes z)^{(1)})^{m-i} \mathbf{1}_\gamma \right) = x^i \quad (0 \leq i \leq m).$$

Theorem 3.8. *For $m \in \mathbb{Z}_{\geq 0}$, we have*

$$T_m \circ \left(G_0^{(\infty)} - \frac{1}{4} h_0(h_0 + 2) + \frac{1}{2} \gamma_1^{(1)} \gamma_1^{(\infty)} \right) = \frac{t}{\hbar^2} \widehat{H}_{III} \left(x, \hbar \frac{\partial}{\partial x}, m\hbar, b, t \right) \circ T_m$$

as linear maps from M_m to $\bigoplus_{i=0}^m \mathbb{C}x^i$ with

$$\gamma_0^{(1)} = 2(m-1) - \frac{b}{\hbar}, \quad \gamma_1^{(1)} = \frac{1}{\hbar}, \quad \gamma_1^{(\infty)} = -\frac{t}{\hbar}.$$

Proof. It follows from direct computations. \square

As in the cases above, the Schrödinger equation for the quantum third Painlevé equation has the following symmetry.

Proposition 3.9. *If a function $\Phi^{\text{III}}(x, t, a, b, \hbar)$ is a solution to the Schrödinger equation for the quantum third Painlevé equation, then the function $\Psi^{\text{III}}(x, t, a, b, \hbar)$ defined as*

$$\Psi^{\text{III}}(x, t, a, b, \hbar) = \Phi^{\text{III}}(-x, t, -a, -b, \hbar) \quad (3.19)$$

is a solution to

$$\hbar \frac{\partial}{\partial t} \Psi^{\text{III}}(x, t) = \widehat{H}'_{\text{III}} \Psi^{\text{III}}(x, t),$$

where $\widehat{H}'_{\text{III}}$ are obtained by replacing \hbar with $-\hbar$ in \widehat{H}_{III} .

Proof. It follows from direct computations. \square

3.5. The case of $J = \text{II}$. Let $n = 0$ and $r_\infty = 3$. Then $M = M'(\gamma_1^{(\infty)}, \gamma_2^{(\infty)}, \gamma_3^{(\infty)})$ and the confluent KZ equation for an unknown function $\Psi(\gamma_1^{(\infty)})$ taking values in M is

$$\kappa \gamma_3^{(\infty)} \frac{\partial}{\partial \gamma_1^{(\infty)}} \Psi(\gamma_1^{(\infty)}) = \left(G_2^{(\infty)} - \frac{1}{4} \left(\gamma_1^{(\infty)} \right)^2 - \frac{1}{2} \gamma_2^{(\infty)} + \frac{1}{2} h^{(\infty)} \gamma_2^{(\infty)} \right) \Psi(\gamma_1^{(\infty)}), \quad (3.20)$$

where $\kappa \in \mathbb{C}$ and

$$G_2^{(\infty)} = \frac{1}{2} \left((e \otimes z)^{(\infty)} (f \otimes z)^{(\infty)} + (f \otimes z)^{(\infty)} (e \otimes z)^{(\infty)} + \frac{1}{2} ((h \otimes z)^{(\infty)})^2 \right).$$

Let M_m ($m \in \mathbb{Z}_{\geq 0}$) be the weight space of M with the weight $-2m$, namely,

$$\begin{aligned} M_m &= \left\{ v \in M \mid h^{(\infty)}(v) = -2mv \right\} \\ &= \bigoplus_{i=0}^m \mathbb{C}((f \otimes z)^{(\infty)})^i ((f \otimes z^2)^{(\infty)})^{m-i} \mathbf{1}_\gamma. \end{aligned}$$

An integral formula taking values in M_m for a solution to the confluent KZ equation (3.20) is

$$\begin{aligned} &\int_\Gamma \prod_{i=1}^m du_i \prod_{1 \leq i < j \leq m} (u_i - u_j)^{2/\kappa} \exp \left(\frac{1}{\kappa} \sum_{i=1}^m \left(u_i \gamma_1^{(\infty)} + \frac{u_i^2}{2} \gamma_2^{(\infty)} + \frac{u_i^3}{3} \gamma_3^{(\infty)} \right) \right) \\ &\quad \times \prod_{i=1}^m ((f \otimes z)^{(\infty)} + (f \otimes z^2)^{(\infty)} u_i) \mathbf{1}_\gamma, \end{aligned} \quad (3.21)$$

where Γ is an appropriate cycle (see Proposition 4.2 in [7]).

We define linear isomorphisms $T_m : M_m \rightarrow \bigoplus_{i=0}^m \mathbb{C}x^i$ ($m \in \mathbb{Z}_{\geq 0}$) as

$$T_m(((f \otimes z)^{(\infty)})^i ((f \otimes z^2)^{(\infty)})^{m-i} \mathbf{1}_\gamma) = x^i \quad (0 \leq i \leq m).$$

Theorem 3.10. *For $m \in \mathbb{Z}_{\geq 0}$, we have*

$$T_m \circ \left(G_2^{(\infty)} - \frac{1}{4} \left(\gamma_1^{(\infty)} \right)^2 - \frac{1}{2} \gamma_2^{(\infty)} + \frac{1}{2} h^{(\infty)} \gamma_2^{(\infty)} \right) = \frac{2}{\hbar^2} \widehat{H}_{\text{II}} \circ T_m,$$

as linear maps from M_m to $\bigoplus_{i=0}^m \mathbb{C} x^i$ with

$$\gamma_1^{(\infty)} = \frac{t}{\hbar}, \quad \gamma_2^{(\infty)} = 0, \quad \gamma_3^{(\infty)} = \frac{2}{\hbar}.$$

Proof. It follows from direct computations. □

As in the cases above, the Schrödinger equation for the quantum second Painlevé equation has the following symmetry.

Proposition 3.11. *If a function $\Phi^{\text{II}}(x, t, a, \hbar)$ is a solution to the Schrödinger equation for the quantum second Painlevé equation, then the function $\Psi^{\text{II}}(x, t, a, \hbar)$ defined as*

$$\Psi^{\text{II}}(x, t, a, \hbar) = \Phi^{\text{II}}(-x, t, -a, \hbar) \tag{3.22}$$

is a solution to

$$\hbar \frac{\partial}{\partial t} \Psi^{\text{II}}(x, t) = \widehat{H}'_{\text{II}} \Psi^{\text{II}}(x, t),$$

where \widehat{H}'_{II} are obtained by replacing \hbar with $-\hbar$ in \widehat{H}_{IV} .

Proof. It follows from direct computations. □

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